

# SYMMETRY OF TRAVELING WAVE SOLUTIONS TO THE ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

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**ABSTRACT.** In this paper, we prove even symmetry of monotone traveling wave solutions to the balanced Allen-Cahn equation in the entire plane. Related results for the unbalanced Allen-Cahn equation are also discussed.

**Keywords:** Allen-Cahn equation, Hamiltonian Identity, Level Set, Symmetry, Traveling Wave Solution, Mean Curvature Soliton

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## 1. INTRODUCTION

The study of traveling wave solutions is a classic area of research of reaction diffusion equations. In the last decade, traveling wave solutions and generalized traveling wave solutions have generated a lot of excitements among mathematicians, partially due to rich phenomena in various branches of applied sciences which are related to traveling fronts, such as flame propagation in various media, population spreading, etc; The research is also fueled by new discoveries of deep and beautiful mathematics related to traveling waves. See, for example, a recent survey [7] and a monograph [8] for details. In this paper, we are mainly concerned with traveling wave solutions in the entire plane of the Allen-Cahn equation with a balanced double well potential, even though we also discuss Allen-Cahn equation with an unbalanced potential or in the entire higher dimensional space. Namely, we consider a traveling wave solution  $v(x, y, t) = u(x, y - ct)$  of the Allen-Cahn equation

$$(1.1) \quad v_t = \Delta_x v + v_{yy} - F'(v), \quad (x, y, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$$

where  $c > 0$  and  $F$  is a double-well potential, i.e.,  $F$  is  $C^3$  and satisfies

$$(1.2) \quad \begin{cases} F'(-1) = F'(1) = 0, & F''(-1) > 0, F''(1) > 0 \\ F'(s) > 0, s \in (-1, \theta); & F'(s) < 0, s \in (\theta, 1) \end{cases}$$

for some  $\theta \in (0, 1)$ . Without loss of generality, we may assume that  $F(-1) = 0$  and  $\theta = 0$ . If  $F(1) = F(-1) = 0$ ,  $F$  is called a balanced double well potential. Otherwise, it is called an unbalanced double well potential, and in this case we may assume that  $F(1) > F(-1) = 0$  without loss of generality.

A typical example of balanced double well potential is  $F(u) = \frac{1}{4}(1 - u^2)^2$ ,  $u \in \mathbb{R}$ , while a typical unbalanced double well potential is  $F(u) = \frac{1}{4}(1 - u^2)^2 - a(u^3/3 - u)$  with  $a \in (-1, 0)$ . Note that  $F'(u) = (u - a)(u^2 - 1)$  in the latter case.

The value of  $u(x, y)$  may be restricted to  $[-1, 1]$ . It is obvious that  $u$  satisfies an elliptic equation

$$(1.3) \quad \Delta_x u + u_{yy} + cu_y - F'(u) = 0, \quad |u| \leq 1, \quad (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

We may assume that the traveling wave solution is monotone in time and hence in the direction of  $y$ . Without loss of generality, we assume

$$(1.4) \quad u_y(x, y) > 0, \quad (x, y) \in \mathbb{R}^n$$

We may also assume that the solution  $u$  connects two stable states, i.e.,

$$(1.5) \quad \lim_{y \rightarrow \pm\infty} u(x, y) = \pm 1, \quad x \in \mathbb{R}^{n-1}.$$

We note that the limit condition above does not need to be uniform in  $x$ . Indeed, we shall see that the limits are not uniform. When  $n = 1$  there exists a unique speed  $c_0 \geq 0$  such that (1.3) has a unique solution  $g(y)$  (up to translation) satisfying the monotone condition (1.4), i.e.,

$$(1.6) \quad \begin{cases} g''(s) + c_0 g'(s) - F'(g(s)) = 0, & s \in \mathbb{R}, \\ \lim_{s \rightarrow \infty} g(s) = 1, & \lim_{s \rightarrow -\infty} g(s) = -1. \end{cases}$$

where  $c_0 = 0$  in the balanced case and  $c_0 > 0$  in the unbalanced case. We may assume that  $g(0) = 0$ . The solution  $g$  is non-degenerate in the sense that the linearized operator has a kernel spanned only by  $g'$ .

It is well-known that when  $F$  is balanced,  $g$  is a minimizer of the following energy functional

$$\mathbf{E}(v) := \int_{-\infty}^{\infty} \left[ \frac{1}{2} |v'|^2 + F(v) \right] dx$$

in  $\mathcal{H} := \{v \in H_{loc}^1(\mathbb{R}) : -1 \leq v \leq 1, \lim_{s \rightarrow \pm\infty} v(s) = \pm 1\}$  and

$$\mathbf{e} := \mathbf{E}(g) = \int_{-1}^1 \sqrt{2F(u)} du < \infty.$$

There is a significant difference between the balanced and the unbalanced Allen-Cahn equation when traveling wave solutions are concerned. The difference of zero speed and positive speed of one dimensional traveling wave solution  $g$  for the balanced and unbalanced potential leads to a fundamental difference of the structure of traveling fronts in higher dimensional spaces, as discussed below, as well as shown in Theorem 1.1 and Theorem 3.2 and [24]. The existence, uniqueness, stability and other qualitative properties of traveling wave solutions to the unbalanced Allen-Cahn equation have been studied in [30] [31], [42], [43], [45], [46]. Similar traveling wave solutions for Fisher-KPP type equation or combustion equation have also been investigated in [11], [28], [32], [41]. The typical shape of traveling fronts studied in these articles are conical. The stability and uniqueness results are also based on the assumption that the traveling fronts are conical. In particular, the traveling fronts for these equations are globally Lipschitz continuous. Traveling wave solutions for the balanced Allen-Cahn equation are first studied in [14], where non-conical and non-planar traveling fronts with axial symmetry are proven existing. It is noted that the traveling fronts are not globally Lipschitz. Indeed, the following theorem is proven in [14].

**Theorem A** (Chen, Guo, Hamel, Ninomiya, Roquejoffre, 2007). *For any  $c > 0$ , there exists a solution  $U(x, y) = U(|x|, y)$  to (1.3), (1.4), (1.5) such that  $U_r(r, y) < 0$  for  $r > 0$  and  $U(0, 0) = 0$ . Furthermore, if the 0-level set of  $U$  is denoted by  $\Gamma$ , then*

(i) *when  $n = 2$ ,  $\Gamma$  is asymptotically a hyperbolic cosine curve, i.e., for some  $A > 0$*

$$(1.7) \quad \lim_{y \rightarrow \infty, U(x, y) = 0} \frac{\cosh(2\mu x)}{\mu y} = \frac{A}{c}$$

where  $\mu = \sqrt{F''(1)}$ .

(ii) *when  $n > 2$ ,  $\Gamma$  is asymptotically a paraboloid, i.e.,*

$$(1.8) \quad \lim_{y \rightarrow \infty, U(x, y) = 0} \frac{|x|^2}{2y} = \frac{n-2}{c}.$$

It is very interesting to note that for  $n > 2$  the traveling fronts are very similar to the translating radial solutions to the mean curvature flow, i.e., the entire radial solutions to

$$(1.9) \quad \operatorname{div}\left(\frac{D\Gamma}{\sqrt{1+|D\Gamma|^2}}\right) = \frac{1}{\sqrt{1+|D\Gamma|^2}} \quad \text{in } \mathbb{R}^{n-1},$$

where  $y = \Gamma(|x|)$  can be computed as

$$(1.10) \quad \Gamma(r) = \frac{r^2}{2(n-2)} - \ln r + C_1 - \frac{(n-2)(n-5)}{2}r^{-2} + o(r^{-2}).$$

See [3], [26]. This is not surprising due to the connection between the surface motion by mean curvature and the interface motion of solutions to the balanced Allen-Cahn equation. See, for example, [13], [48], [17], [35]. It is reasonable to expect that the traveling fronts with unit speed  $c = 1$  should be related to the translating mean curvature flow of unit speed. The case  $n = 2$  is slightly different, in this case the solution for the translating mean curvature flow is the “grim reaper”, i.e., the curve given by  $\Gamma(x) = \log \sec(x)$ , while the traveling front is a hyperbolic cosine. The discrepancy between these two curves is due to the strong interaction caused by the reaction term in the Allen-Cahn equation.

Recent studies on the translating mean curvature flow reveal very interesting properties of convex solutions. See [51], [50], [26] and references therein. In particular, it is shown in [50] that convex solutions to (1.9) must be rotationally symmetric for  $n \leq 3$ . It is then natural to ask whether a traveling wave solution to (1.1) with monotone (1.4) and limit condition (1.5) must be rotationally symmetric, or, in the terminology of this paper, axially symmetric after a proper translation in  $x$  variable. In this paper, we shall show that this is indeed true for  $n = 2$ . To be more precise, we have the following main theorem.

**Theorem 1.1.** *Assume that  $F$  is a balanced double well potential satisfying (1.2) and  $F(-1) = F(1) = 0$ . Suppose  $u$  satisfies (1.3), (1.4) and (1.5). Then, when  $n = 2$ ,  $u$  is evenly symmetric with respect to  $x$  after a proper translation, and  $u_x(x, y) < 0$  for  $x > 0$ .*

In dimensions  $n \geq 3$ , obviously we need more conditions, since if  $u(x, y)$  is a solution in  $\mathbb{R}^n$ , then a trivial extension  $u(x, s, y) = u(x, y)$  is a solution in  $\mathbb{R}^{n+1}$ . It remains open whether all monotone traveling wave solutions with the limit condition

(1.5) must be either axially symmetric or trivial extension of a axially symmetric solution in lower dimensional space. Due to possible existence of non-rotationally convex translating mean curvature flow in higher dimensions ([51], [50]), the answer for the above question is probably not affirmative except for  $n = 3$ . The latter will be discussed in a forthcoming paper [24]. We note that symmetry results have also been proven for certain saddle solutions of Allen-Cahn equation (1.11) in [25] and for solutions of nonlinear stationary Schrodinger equation in [27].

A very closely related question is the De Giorgi conjecture, which may be regarded as assertion on the one dimensional symmetry of solutions to (1.3) when  $c = 0$ , i.e.,

$$(1.11) \quad \Delta u - F'(u) = 0, \quad |u| < 1, \quad (x, y) \in \mathbb{R}^n.$$

The conjecture may be stated as follows.

**Conjecture.** (*De Giorgi, 78*) *If  $u$  satisfies (1.11) and (1.4), then for at least  $n \leq 8$ ,  $u$  must be a one dimensional solution, i.e. a proper trivial extension, rotation and translation of  $g$ . In other words, the level sets of  $u$  must be hyper planes.*

This conjecture is based on the famous Bernstein problem regarding the classification of complete minimal graph in  $\mathbb{R}^n$  ([10], [21]). The De Giorgi conjecture is proven affirmatively for  $n = 2$  in [20] and for  $n = 3$  in [4]. With the extra limit condition (1.5), it is proven for  $n \leq 8$  in [47]. Recently, non planar solutions for (1.11) with  $n \geq 9$  are constructed in [16] by using the non-planar minimal graph by Bombieri, De Giorgi and Giusti ([10], [21]).

For the case of an unbalanced double well potential, we shall show a similar result as Theorem 1.1, which improves a classification theorem of (??) for all monotone traveling wave solutions in  $\mathbb{R}^2$ . See Theorem 3.2. in Section 3.

The paper is organized as follows. In Section 2, the main result Theorem 1.1 shall be proved. Theorem 3.2., the classification result for traveling wave solutions of the unbalanced Allen-Cahn equation in  $\mathbb{R}^2$ , will be proved in Section 3. Finally, traveling waves solutions connecting various stationary one dimensional solutions will be investigated in Section 4.

## 2. EVEN SYMMETRY OF TRAVELING WAVE SOLUTIONS OF THE BALANCED ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

Through out in this section, we assume that  $n = 2$  and the double well potential  $F$  is balanced, i.e.,  $F(-1) = F(1) = 0$ . We shall prove Theorem 1.1 in three main steps. First we carry out a preliminary asymptotical analysis of the level sets of the solution  $u$  and show that the slope of the 0-level curve  $y = \gamma(x)$  must tend to  $\pm\infty$  as  $x$  tends to  $\pm\infty$ . Second, we show that  $y = \gamma(x)$  is asymptotically hyperbolic cosine and obtain a very detailed asymptotical formula. Last, we complete the proof by using the asymptotical formula of the level curve and the moving plane method. We note that the regularity condition of  $F$  can be replaced by  $C^{2,\beta}$  with some  $\beta \in (0, 1)$  for most discussion below, except in (2.20) where the third derivatives of  $\phi$  with respect to  $l_i$  require  $F \in C^3$ .

**2.1. Preliminary Analysis of the level set.** We first show an important lemma which asserts the integrability of  $u_y$ .

**Lemma 2.1.** *Suppose that  $u$  is a solution to (1.3), (1.4) and (1.5). Then*

$$(2.1) \quad \int_{\mathbb{R}^2} u_y^2 dx dy < \infty.$$

*Proof.* Define

$$h(x) = \int_{\mathbb{R}} u_x u_y dy, \quad x \in \mathbb{R}.$$

Since  $u$  is bounded in  $C^3(\mathbb{R}^n)$  by the standard elliptic estimates and  $u_y$  is positive, it is easy to see that  $h(x)$  is well-defined and

$$|h(x)| < C, \quad x \in \mathbb{R}$$

for some constant  $C > 0$ .

Note that due to (1.5), we have

$$\lim_{y \rightarrow \pm\infty} u_x = 0, \quad \lim_{y \rightarrow \pm\infty} u_y = 0, \quad x \in \mathbb{R}.$$

Differentiating  $h(x)$  with respect to  $x$  and using the equation, we obtain

$$(2.2) \quad \begin{aligned} h'(x) &= \int_{\mathbb{R}} (u_{xx} u_y + u_x u_{xy}) dy \\ &= \int_{\mathbb{R}} \left[ \frac{\partial}{\partial y} \left( F(u) - \frac{1}{2} u_y^2 + \frac{1}{2} u_x^2 \right) - c u_y^2 \right] dy \\ &= -c \int_{\mathbb{R}} u_y^2 dy. \end{aligned}$$

Then

$$(2.3) \quad \int_a^b \int_{\mathbb{R}} u_y^2 dy dx = \frac{1}{c} (h(a) - h(b)).$$

The bound of  $h(x)$  immediately leads to the integrability of  $u_y^2$  in  $\mathbb{R}^2$ .  $\square$

Due to (1.4) and (1.5), the 0-level set of  $u$  is a  $C^3$  graph of a function defined in  $\mathbb{R}$ . We let  $y = \gamma(x)$ ,  $x \in \mathbb{R}$  be such a function. The next lemma asserts that the slope of  $y = \gamma(x)$  must tend to infinity as  $x$  goes to infinity.

**Lemma 2.2.** *There holds*

$$(2.4) \quad \lim_{|x| \rightarrow \infty} |\gamma'(x)| = \infty.$$

*Proof.* Since  $u$  is bounded in  $C^3(\mathbb{R}^2)$ , Lemma 2.1 implies that

$$\lim_{|x| \rightarrow \infty} u_y(x, y) = 0, \quad \text{uniformly in } y \in \mathbb{R}.$$

Now assume that (2.4) is not true, then there exists a sequence  $\{x_m\}$  such that  $|x_m|$  goes to infinity and

$$\lim_{m \rightarrow \infty} \gamma'(x_m) = k_0$$

for some constant  $k_0$ .

We shall translate  $u$  along this sequence of  $x_m$ . Define

$$u_m(x, y) = u(x + x_m, y + \gamma(x_m)), \quad (x, y) \in \mathbb{R}^2.$$

By the standard theory for elliptic equations, we know that  $u_m$  is bounded in  $C^{3,\beta}(\mathbb{R}^2)$ . Then there is a subsequence, which we still denote by  $\{x_m\}$ , such that  $u_m$  converges to a function  $u_*$  in  $C_{loc}^3(\mathbb{R}^2)$ . It is easy to see that  $u_*(0, 0) =$

0,  $\frac{\partial}{\partial y}u_*(x, y) = 0$ ,  $(x, y) \in \mathbb{R}^2$ . Then  $u_*(x, y) = g_*(x)$  for some  $C^3$  function  $g_*$  which is a solution to the one dimensional stationary Allen-Cahn equation

$$(2.5) \quad u_{xx} - F'(u) = 0, \quad x \in \mathbb{R}.$$

Furthermore, since  $u(x, \gamma(x)) = 0$  and hence

$$u_x(x, \gamma(x)) + u_y(x, \gamma(x))\gamma'(x) = 0, \quad x \in \mathbb{R},$$

we obtain

$$g'_*(0) = \lim_{m \rightarrow \infty} u_x(x_m, \gamma(x_m)) = - \lim_{m \rightarrow \infty} u_y(x_m, \gamma(x_m))\gamma'(x_m) = 0.$$

Then we conclude that  $g_* \equiv 0$ . We claim that this will lead to a contradiction.

As in the proof of Lemma 2.1, we define

$$h_m(x) = \int_{-\infty}^0 \frac{\partial u_m}{\partial x} \frac{\partial u_m}{\partial y} dy.$$

It is easy to see that  $|h_m(x)| < C$  for some constant independent of both  $x$  and  $m$ . We can also derive

$$h'_m(x) = -c \int_{-\infty}^0 \left( \frac{\partial u_m}{\partial y} \right)^2 dy + \frac{1}{2} \left( \frac{\partial u_m}{\partial x} \right)^2(x, 0) - \frac{1}{2} \left( \frac{\partial u_m}{\partial y} \right)^2(x, 0) + F(u_m(x, 0)), \quad x \in \mathbb{R}$$

For any fix  $R > 0$ , in view of (2.1) we have

$$\int_{-R}^R \left[ \frac{1}{2} \left( \frac{\partial u_m}{\partial x} \right)^2(x, 0) - \frac{1}{2} \left( \frac{\partial u_m}{\partial y} \right)^2(x, 0) + F(u_m(x, 0)) \right] dx < C$$

for some constant  $C$  independent of  $m, R$ .

Letting  $m$  go to infinity, we obtain  $2F(0)R \leq C$ , which is a contradiction. The proof of the lemma is then complete.  $\square$

Indeed, we conclude that the level curve must be of one of the following four possibilities:

- (i)  $\lim_{x \rightarrow \infty} \gamma'(x) = +\infty, \quad \lim_{x \rightarrow -\infty} \gamma'(x) = -\infty;$
- (ii)  $\lim_{x \rightarrow \infty} \gamma'(x) = +\infty, \quad \lim_{x \rightarrow -\infty} \gamma'(x) = +\infty;$
- (iii)  $\lim_{x \rightarrow \infty} \gamma'(x) = -\infty, \quad \lim_{x \rightarrow -\infty} \gamma'(x) = -\infty;$
- (iv)  $\lim_{x \rightarrow \infty} \gamma'(x) = -\infty, \quad \lim_{x \rightarrow -\infty} \gamma'(x) = +\infty.$

Moreover, it can also be concluded by the arguments above that the profile of  $u$  along the level curve must be approximately the one dimensional transition layer  $g(x)$  or  $g(-x)$ . To be more precise, we define

$$u_s(x, y) := u(s + x, \gamma(s) + y), \quad (x, y) \in \mathbb{R}^2.$$

The following lemma holds.

**Lemma 2.3.** *The translated solution  $u_s(x, y)$  converges in  $C_{loc}^3(\mathbb{R}^2)$  to either  $g(x)$  or  $g(-x)$  as  $|s|$  tends to infinity.*

**2.2. The exponential decay of  $u$  and the Hamiltonian identity.** In this subsection, we shall show that solution  $u$  must decay exponentially to  $\pm 1$  as the distance from the level set  $y = \gamma(x)$  tends to infinity. The exponential decay of  $u$  will be used to prove a version of Hamiltonian identity for equation (1.3). This type of analysis was first carried out in [14] for the axially symmetric traveling wave solutions. Their arguments are slightly modified and presented here for the convenience of the reader.

Due to the double well potential condition of  $F$ , there exist two constants  $\alpha^+, \alpha^-$  such that  $-1 < \alpha^- < 0 < \alpha^+ < 1$  and

$$F''(s) > \mu_0 > 0, \quad s \in [-1, \alpha^-] \cup [\alpha^+, 1].$$

for some constant  $\mu_0 > 0$ .

Define

$$\begin{aligned} \Omega^+ &:= \{(x, y) \in \mathbb{R}^2 : u(x, y) \geq \alpha^+\}, \quad \Omega^- := \{(x, y) \in \mathbb{R}^2 : u(x, y) \leq \alpha^-\}, \\ \Omega^0 &:= \{(x, y) \in \mathbb{R}^2 : \alpha^- \leq u(x, y) \leq \alpha^+\}, \quad \Omega_y^0 := \{x \in \mathbb{R} : \alpha^- \leq u(x, y) \leq \alpha^+\}, \\ \gamma^\alpha &:= \{(x, y) \in \mathbb{R}^2 : y = \gamma^\alpha(x), u(x, \gamma^\alpha(x)) = \alpha\}, \quad \alpha \in (-1, 1). \end{aligned}$$

By Lemmas 2.2 and 2.3, it is easy to see that  $\text{meas}(\Omega_y^0) < K < \infty$  for some constant  $K$  independent of  $y$ . Indeed, there exists a positive constant  $Y_0 > 0$  and two  $C^3$  functions  $x = k_i(y), i = 1, 2$  such that  $\gamma^0 \cap \{(x, y) \in \mathbb{R}^2 : |y| > Y_0\}$  can be expressed as the graph of  $k_i(y)$ , i.e.,

$$\gamma^0 \cap \{(x, y) \in \mathbb{R}^2 : |y| > Y_0\} = \{(x, y) : x = k_i(y), |y| > Y_0, i = 1, 2\}.$$

In Case (i), both  $k_1$  and  $k_2$  are defined for  $Y > Y_0$ , while in Case (iv),  $k_1$  and  $k_2$  are defined for  $Y < -Y_0$ . We may assume that  $k_1(y) < k_2(y)$  in these two cases.

In Case (ii) and (iii),  $k_1$  is defined for  $y > Y_0$  and  $k_2$  is defined for  $y < -Y_0$ .

In all cases, we have

$$(2.6) \quad |x - k_1(y)| < K, \text{ or } |x - k_2(y)| < K, \quad \forall x \in \Omega_y^0, |y| > Y_0.$$

Now we can state the exponential decay of  $u$  as the following lemma.

**Lemma 2.4.** *There exist constants  $C$  and  $\nu > 0$  such that*

$$(2.7) \quad \begin{cases} |u^2 - 1| + |\nabla u| + |\nabla^2 u| \leq C e^{-\nu d(x, y)}, & |y| > Y_0 \\ |u^2 - 1| + |\nabla u| + |\nabla^2 u| \leq C e^{-\nu |x|}, & |y| \leq Y_0. \end{cases}$$

where  $d(x, y) := \min\{|x - k_1(y)|, |x - k_2(y)|\}$  for  $|y| > Y_0$ .

*Proof.* Let

$$w(x, y) = 1 \mp u(x, y) > 0, \quad (x, y) \in \Omega^\pm.$$

Then, by the definition of  $\mu_0$  and  $\Omega^\pm$ , it is easy to see that

$$w_{xx} + w_{yy} + cw_y - \mu_0 w = \left( \frac{F'(\pm 1) - F'(u)}{\pm 1 - u} - \mu_0 \right) \cdot w \geq 0, \quad (x, y) \in \Omega^\pm.$$

Now we choose two positive constants  $\mu_1$  and  $\mu_2$  as

$$\mu_1 = \frac{c + \sqrt{c^2 + 8\mu_0}}{4}, \quad \mu_2 = \frac{-c + \sqrt{c^2 + 8\mu_0}}{4}.$$

Note that

$$\mu_1 = \mu_2 + c/2, \quad \mu_1^2 + \mu_2^2 = c^2/4 + \mu_0.$$

For any rectangular domain  $D_R := \{(x, y) : |x| \leq R, 0 < y < R\}$ , we consider the function

$$B(x, y) = 4e^{-\mu_2 R - cy/2} \cosh(\mu_1 y) \cosh(\mu_2 x), \quad (x, y) \in D_R.$$

Straight forward computations reveal that

$$B_{xx} + B_{yy} + cB_y - \mu_0 B = 0, \quad \text{in } D_R, \quad B \geq 1 \text{ on } \partial D_R.$$

Now for any  $(x_0, y_0) \in \Omega^\pm$ , let  $R = R(x_0, y_0)$  be the distance of  $(x_0, y_0)$  to  $\Omega^0$  and compare  $w(x, y)$  with  $B(x - x_0, y - y_0)$  in  $D_R(x_0, y_0) := \{(x, y) : (x - x_0, y - y_0) \in D_R\}$ . Then the maximum principle implies that

$$w(x, y) \leq B(x - x_0, y - y_0), \quad (x, y) \in D_R(x_0, y_0).$$

In particular, we have  $w(x_0, y_0) \leq B(0, 0) = 4e^{-\mu_2 R}$ . In view of (2.4), (2.6) and the definition of  $k_1, k_2$ , we know that, for  $R(x, y) \geq K$ , there exists some constant  $\mu_3 \in (0, 1)$  such that  $R(x, y) \geq \mu_3 d(x, y)$  when  $|y| > Y_0$  and  $R(x, y) \geq \mu_3 |x|$  when  $|y| \leq Y_0$ .

Hence we derive

$$|u^2 - 1| \leq C_0 e^{-\nu d(x, y)}, \quad |y| > Y_0; \quad |u^2 - 1| \leq C_0 e^{-\nu |x|}, \quad |y| \leq Y_0.$$

for  $\nu = \mu_2 \mu_3$  and some constant  $C_0 > 0$ . Note that  $\nu < \sqrt{\mu_0} \leq \min\{\sqrt{F''(1)}, \sqrt{F''(-1)}\}$ .

Then (2.7) follows from the standard estimates for elliptic equations.  $\square$

With the exponential decay of  $u$ , we can define

$$(2.8) \quad \rho(y) = \rho(y; u) := \int_{\mathbb{R}} \left[ \frac{1}{2} (|\nabla_x u|^2 - u_y^2) + F(u) \right] dx, \quad y \in \mathbb{R}.$$

The following Hamiltonian identity holds.

**Lemma 2.5.** *For any  $y_0, y \in \mathbb{R}$ , there holds the following Hamiltonian identity*

$$(2.9) \quad \rho(y) - \rho(y_0) = c \int_{y_0}^y \int_{\mathbb{R}} |u_y|^2 dx dy.$$

**2.3. Only Case (i) is valid.** Using the exponential decay (2.7), the Hamiltonian identity (2.9) and Lemma 2.3, we can exclude the Cases (ii)-(iv) in subsection 2.1.

The next lemma further asserts that only the first case is possible.

**Lemma 2.6.** *Assume that  $u$  is solution to (1.3), (1.4) and (1.5), and the graph of  $y = \gamma(x)$  is the 0-level set of  $u$ . Then*

$$(2.10) \quad \lim_{x \rightarrow \infty} \gamma'(x) = \infty, \quad \lim_{x \rightarrow -\infty} \gamma'(x) = -\infty.$$

*Proof.* In Case (ii), using the exponential decay (2.7) and Lemma 2.3, we can compute straight forwardly

$$\lim_{y \rightarrow \infty} \rho(y) = \lim_{s \rightarrow \infty} \rho(0; u_s) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (|g'|^2(x) + F(g(x))) \right] dx = \mathbf{e}$$

and

$$\lim_{y_0 \rightarrow -\infty} \rho(y_0) = \lim_{s \rightarrow -\infty} \rho(0; u_s) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (|g'|^2(x) + F(g(x))) \right] dx = \mathbf{e}.$$



Then the Hamiltonian identity (2.9) leads to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u_y|^2 dx dy = 0.$$

This is a contradiction. Case (iii) can be excluded similarly.

In Case (iv), we have

$$\lim_{y \rightarrow \infty} \rho(y) = 0$$

and

$$\begin{aligned} \lim_{y_0 \rightarrow -\infty} \rho(y_0) &= \lim_{y_0 \rightarrow -\infty} \int_{-\infty}^0 \left[ \frac{1}{2} (|\nabla_x u|^2 - u_y^2) + F(u) \right] dx \\ &\quad + \lim_{y_0 \rightarrow -\infty} \int_0^{\infty} \left[ \frac{1}{2} (|\nabla_x u|^2 - u_y^2) + F(u) \right] dx \\ &= 2 \int_{-\infty}^{\infty} \left[ \frac{1}{2} (|g'|^2(x) + F(g(x))) \right] dx = 2e. \end{aligned}$$

This leads to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u_y|^2 dx dy = -\frac{2e}{c} < 0.$$

This is a contradiction, and the lemma is proven.  $\square$

**2.4. The level set curve is asymptotically hyperbolic cosine.** In this subsection, we shall show that the 0-level set  $y = \gamma(x)$  of  $u$  is asymptotically hyperbolic cosine. It is more convenient to write the level set as the graph of functions  $x = k_1(y)$ ,  $x = k_2(y)$  for  $y > Y_0$  and show that they are logarithmic. In the previous subsection, we have already derived properties for  $y = \gamma(x)$  which can be rewritten for  $x = k_i(y)$  as follows

$$(2.11) \quad \begin{cases} k'_1(y) < 0, & k'_2(y) > 0, & \text{for } y > y_0 \\ \lim_{y \rightarrow \infty} k_1(y) = -\infty, & \lim_{y \rightarrow \infty} k_2(y) = \infty \\ \lim_{y \rightarrow \infty} k'_1(y) = \lim_{y \rightarrow \infty} k'_2(y) = 0 \end{cases}$$

We shall prove the following asymptotical formulas for  $k_i(y)$ ,  $i = 1, 2$ .

**Lemma 2.7.** *There holds*

$$(2.12) \quad \begin{cases} k_1(y) = -\frac{1}{2\mu} \ln(y) + C_1 + o(1), & \text{as } y \rightarrow \infty \\ k_2(y) = \frac{1}{2\mu} \ln(y) + C_2 + o(1), & \text{as } y \rightarrow \infty \end{cases}$$

for some constants  $C_1, C_2$ , where  $\mu = \sqrt{F''(1)} > 0$ .

2.4.1. *A standard profile with two transition layers.* The proof of Lemma 2.7 follows the main ideas of [14] in the derivation of similar formula for axially symmetric traveling wave solutions. Instead of only dealing with one unknown function in [14], here we need to consider the coupled functions  $k_i(y)$ ,  $i = 1, 2$ . We shall approximate  $u(x, y)$  as functions of  $x$  by a family of standard profiles of two transition layers for  $y$  sufficiently large. Namely, for  $l_1 < l_2$  and  $2l = l_2 - l_1$  sufficiently large, we define a continuous and piecewise smooth function  $\phi = \phi(l_1, l_2, x)$  so that it is the solution of one dimensional Allen-Cahn equation in three segments of  $\mathbb{R}$ :

$$(2.13) \quad \begin{cases} \phi'' - F'(\phi) = 0, & x \in (-\infty, l_1) \cup (l_1, l_2) \cup (l_2, \infty) \\ \phi(x) > 0, & x \in (l_1, l_2); \quad \phi(x) < 0, & x \in (-\infty, l_1) \cup (l_2, \infty) \\ \phi(l_1) = \phi(l_2) = 0, & \lim_{x \rightarrow \pm\infty} \phi(x) = -1 \end{cases}$$

Below we collect some basic facts about  $\phi = \phi(l_1, l_2, x)$  and related functions. Indeed,  $\phi(l_1, l_2, x) = g(l_2 - x)$  for  $x > l_2$  and  $\phi(l_1, l_2, x) = g(x - l_1)$  for  $x < l_1$ . For  $x \in (l_1, l_2)$ ,  $\phi(l_1, l_2, x) = g(l, x - (l_1 + l_2)/2)$  where  $g(l, x) = g(l, -x)$  can be solved explicitly by

$$\begin{aligned} g_x^2(l, x) &= 2F(g(l, x)) - 2F(g(l, 0)), \quad x \in (-l, l) \\ \int_{g(l, x)}^{g(l, 0)} \frac{ds}{\sqrt{2(F(s) - F(g(l, 0)))}} &= x, \quad x \in (0, l) \end{aligned}$$

where  $0 < g(l, 0) < 1$ .

Note that elementary computations can lead to  $\lim_{l \rightarrow \infty} g(l, 0) = 1$  and

$$l = \int_0^{g(l, 0)} \frac{ds}{\sqrt{2(F(s) - F(g(l, 0)))}} = -\frac{\ln(1 - g(l, 0))}{\mu} + A_1 + o(1)$$

as  $l \rightarrow \infty$ , where  $A_1$  is a constant depending only on  $F$ . It is also easy to see that  $g(l, x)$  is the minimizer of

$$\mathbf{E}_l(v) := \int_{-l}^l \left[ \frac{1}{2} |v'|^2 + F(v) \right] dx$$

in  $\mathcal{H}_l := \{v \in H_0^1([-l, l]) : 0 \leq v \leq 1, v(-l) = v(l) = 0\}$  when  $l$  is sufficiently large.

If we denote

$$E(l) := \mathbf{E}_l(g(l, \cdot)) = \mathbf{E}(\phi(-l, l, \cdot)) - \mathbf{e},$$

then  $E(l) = \mathbf{e} + o(1)$  and

$$(2.14) \quad E_l := \frac{\partial E(l)}{\partial l} = |g'(0)|^2 - g_x^2(l, l) = 2F(g(l, 0)) = 2\mathbf{e}Ae^{-2\mu l + o(1)}$$

where  $A$  is a positive constant depending only on  $F$  (see [14]).

For a piecewise continuous function  $\psi(x)$  with possible jump discontinuities at  $x = l_1, l_2$ , we define

$$\begin{aligned} \hat{\psi} &= \psi(l_1+) - \phi(l_1-), \quad \check{\psi} = \psi(l_2+) - \phi(l_2-), \\ \tilde{\psi} &= \frac{1}{2}(\psi(l_1-) + \phi(l_1+)), \quad \bar{\psi} = \frac{1}{2}(\psi(l_2+) + \phi(l_2-)). \end{aligned}$$

Note that  $E_l = -\hat{\phi}_{l_1}^2 = -\check{\phi}_x^2 = \phi_{l_2}^2 = \tilde{\phi}_x^2$ .

We also use the norm and inner product of  $L^2(\mathbb{R})$ , i.e.,

$$\langle \psi_1, \psi_2 \rangle := \int_{\mathbb{R}} \psi_1 \psi_2 dx, \quad \|\psi\|^2 := \langle \psi, \psi \rangle.$$

Now we state the following lemma.

**Lemma 2.8.** *For  $l = (l_2 - l_1)/2 > 0$ ,  $\phi(l_1, l_2, x)$  is smooth except at  $x = l_1, l_2$  and*

$$\phi_{l_1} \leq 0, \quad \phi_{l_2} \geq 0, \quad \|\phi_{l_i}\|^2 = E(l) + o(1) = \mathbf{e} + o(1), \quad i = 1, 2.$$

*Furthermore, there exists a constant  $C > 0$  such that  $\forall l > 1$*

$$\begin{aligned} & \sum_{i=1,2} \|\phi_{l_i}\|_{L^1(\mathbb{R})} + \sum_{i,j=1,2} \|\phi_{l_i l_j}\|_{L^1(\mathbb{R})} + \sum_{i,j,k=1,2} \|\phi_{l_i l_j l_k}\|_{L^1(\mathbb{R})} \leq C; \\ & \sum_{i=1,2} \|\phi_{l_i}\| + \sum_{i,j=1,2} \|\phi_{l_i l_j}\| + \sum_{i,j,k=1,2} \|\phi_{l_i l_j l_k}\| \leq C \\ & \sum_{i=1,2} (|\hat{\phi}_{l_i}| + |\check{\phi}_{l_i}|) + \sum_{i,j,k=1,2} (|\hat{\phi}_{l_i l_j}| + |\check{\phi}_{l_i l_j}|) + \sum_{i=1,2} (|\hat{\phi}_{x l_i}| + |\check{\phi}_{x l_i}|) \leq C \cdot E_l \\ & | \langle \phi_{l_1}, \phi_{l_2} \rangle | + |E_{II}| + |E_{III}| \leq C \cdot E_l \end{aligned}$$

**2.4.2. Derivation of ordinary differential equations for  $l_i, i = 1, 2$ .** Now, for  $y > Y_1$  sufficiently large, we can choose a unique pair  $l_1(y) < l_2(y)$  so that

$$(2.15) \quad \|u(\cdot, y) - \phi(l_1(y), l_2(y), \cdot)\|_{L^2(\mathbb{R})} = \inf_{l_1 < l_2} \{\|u(\cdot, y) - \phi(l_1, l_2, \cdot)\|\}.$$

As we shall show, the asymptotical behavior of  $u(x, y)$  near  $y = \infty$  can be accurately described by the dynamics of  $l_i(y), i = 1, 2$ . (See, e.g., [14] Section 6.1 for an intuitive explanation for the case  $l_1 = -l_2$  by using invariant manifold and center manifold terminology.)

Let

$$v(x, y) = u(x, y) - \phi(l_1(y), l_2(y), x), \quad x \in \mathbb{R}, \quad y \geq Y_1.$$

In view of Lemma 2.3, Lemma 2.4 and Lemma 2.6, we see that

$$(2.16) \quad k_1(y) - l_1(y) \rightarrow 0, \quad k_2(y) - l_2(y) \rightarrow 0, \quad \|v(\cdot, y)\| \rightarrow 0, \quad \text{as } y \rightarrow \infty.$$

Moreover, using the implicit function theorem, one can see that for  $y > Y_1$  sufficiently large, the functions  $l_i(y), i = 1, 2$  are smooth and satisfies

$$(2.17) \quad l'_1(y) < 0, \quad l'_2(y) > 0, \quad \lim_{y \rightarrow \infty} l'_i(y) = 0, \quad i = 1, 2.$$

(See, e.g., [14] Lemma 6.2 for a similar statement for the case  $l_1 = -l_2$ .)

It is also obvious that

$$(2.18) \quad \lim_{y \rightarrow \infty} \|(v| + |\nabla v|)\|_{L^\infty(\mathbb{R})} = 0.$$

From (2.15) it is easy to see that

$$(2.19) \quad \langle v(\cdot, y), \phi_{l_i}(l_1(y), l_2(y), \cdot) \rangle = 0, \quad i = 1, 2, \quad y \geq Y_1$$

Differentiating the above identities with respect to  $y$  and dropping the variables of functions for the simplicity of notation, we obtain

$$(2.20) \quad \langle v_y, \phi_{l_i} \rangle + \sum_{j=1,2} \langle v, \phi_{l_i l_j} \rangle l'_j - v \hat{\phi}_{l_i} l'_1 + v \check{\phi}_{l_i} l'_2 = 0, \quad i = 1, 2.$$

Differentiating (2.20) for  $i = 1$  with respect to  $y$ , we have

$$\begin{aligned}
& \langle v_{yy}, \phi_{l_1} \rangle + \langle v_y, \phi_{l_1 l_1} \rangle l'_1 + \langle v_y, \phi_{l_1 l_2} \rangle l'_2 - v_y \hat{\phi}_{l_1} l'_1 + v_y \check{\phi}_{l_1} l'_2 \\
& + \langle v_y, \phi_{l_1 l_1} \rangle l'_1 + [\langle v, \phi_{l_1 l_1 l_1} \rangle l'_1 + \langle v, \phi_{l_1 l_1 l_2} \rangle l'_2 - v \hat{\phi}_{l_1 l_1} l'_1 + v \check{\phi}_{l_1 l_1} l'_2] l'_1 + \langle v, \phi_{l_1 l_1} \rangle l''_1 \\
& + \langle v_y, \phi_{l_1 l_2} \rangle l'_2 + [\langle v, \phi_{l_1 l_1 l_2} \rangle l'_1 + \langle v, \phi_{l_1 l_2 l_2} \rangle l'_2 - v \hat{\phi}_{l_1 l_2} l'_1 + v \check{\phi}_{l_1 l_2} l'_2] l'_2 + \langle v, \phi_{l_1 l_2} \rangle l''_2 \\
& - [v_y \hat{\phi}_{l_1} + v \hat{\phi}_{l_1 l_1} \cdot l'_1 + v \hat{\phi}_{l_1 l_2} \cdot l'_2] \cdot l'_1 - v \hat{\phi}_{l_1} l''_1 \\
& + [v_y \check{\phi}_{l_1} + v \check{\phi}_{l_1 l_1} \cdot l'_1 + v \check{\phi}_{l_1 l_2} \cdot l'_2] l'_2 + v \check{\phi}_{l_2} l''_2 = 0.
\end{aligned}$$

This leads to

$$(2.21) \quad |\langle v_{yy}, \phi_{l_1} \rangle| + |\langle v_y, \phi_{l_1} \rangle| = o(1)(|l'_1| + |l'_2| + |l''_1| + |l''_2|), \quad \text{as } y \rightarrow \infty.$$

Similar computations can also be done for  $i = 2$ .

Now, using equation (1.3) we derive

$$(2.22) \quad v_{xx} + v_{yy} + cv_y - ((F'(v + \phi) - F'(\phi)) + \phi_{yy} + c\phi_y) = 0, \quad (x, y) \in \mathbb{R}^2 \setminus \Gamma, y > Y_1.$$

where

$$\begin{aligned}
\phi_y &= \phi_{l_1} \cdot l'_1 + \phi_{l_2} \cdot l'_2 \\
\phi_{yy} &= \phi_{l_1 l_1} (l'_1)^2 + 2\phi_{l_1 l_2} (l'_1 l'_2) + \phi_{l_2 l_2} (l'_2)^2 + \phi_{l_1} \cdot l''_1 + \phi_{l_2} \cdot l''_2.
\end{aligned}$$

Multiplying (2.22) by  $\phi_{l_1}$  and integrating over  $\mathbb{R}$ , we obtain

$$\begin{aligned}
& (cl'_1 + l''_1) \|\phi_{l_1}\|^2 + \langle \phi_{l_1}, \phi_{l_2} \rangle (cl'_2 + l''_2) + \langle \phi_{l_1 l_1}, \phi_{l_1} \rangle (l'_1)^2 \\
& + \langle \phi_{l_1 l_2}, \phi_{l_1} \rangle (l'_1 l'_2) + \langle \phi_{l_2 l_2}, \phi_{l_1} \rangle (l'_2)^2 = J_{1,1} + J_{1,2} - J_{1,3}
\end{aligned}$$

where

$$\begin{aligned}
J_{1,1} &= \langle F''(\phi)v - v_{xx}, \phi_{l_1} \rangle; \\
J_{1,2} &= \langle F'(v + \phi) - F'(\phi) - F''(\phi)v, \phi_{l_1} \rangle; \\
J_{1,3} &= \langle v_{yy} + cv_y, \phi_{l_1} \rangle.
\end{aligned}$$

Using  $E_l = -\hat{\phi}_{l_1}^2 = \check{\phi}_{l_1}^2$ , it can be computed that

$$(2.23) \quad J_{1,1} = v_x \hat{\phi}_{l_1} - v \hat{\phi}_{l_1 x} + v_x \check{\phi}_{l_1} - v \check{\phi}_{l_1 x} = -E_l(1 + O(|v_x| + |v|)) = E_l(1 + o(1)).$$

Here we have used (2.18), the fact

$$\hat{v}_x = -\hat{\phi}_x = \hat{\phi}_{l_1}, \quad \check{v}_x = -\check{\phi}_x = -\check{\phi}_{l_1}$$

and

$$\begin{aligned}
v_x \hat{\phi}_{l_1} &= \hat{v}_x \hat{\phi}_{l_1} + \check{v}_x \hat{\phi}_{l_1} = \frac{1}{2} \hat{\phi}_{l_1}^2 + \check{v}_x \hat{\phi}_{l_1} \\
v_x \check{\phi}_{l_1} &= \check{v}_x \bar{\phi}_{l_1} + \bar{v}_x \check{\phi}_{l_1} = -\frac{1}{2} \check{\phi}_{l_1}^2 + \bar{v}_x \check{\phi}_{l_1}
\end{aligned}$$

On the other hand, we have

$$(2.24) \quad J_{1,2} = O(1) \langle v^2, \phi_{l_1} \rangle.$$

In view of (2.21), (2.23) and (2.24), we obtain

$$\begin{aligned}
(2.25) \quad & cl'_1 + l''_1 + o(1)(cl'_2 + l''_2) = -\frac{E_l}{\mathbf{e}}(1 + o(1)) + o(1)(l'_2 - l'_1) + O(1)\|v\|(|l''_1| + |l''_2|) + O(1) \langle v^2, \phi_{l_1} \rangle.
\end{aligned}$$

Similarly we can obtain

$$(2.26) \quad cl'_2 + l''_2 + o(1)(cl'_1 + l''_1) = \frac{E_l}{e}(1 + o(1)) + o(1)(l'_2 - l'_1) + O(1)\|v\|(|l'_1| + |l'_2|) + O(1)\langle v^2, \phi_{l_2} \rangle.$$

Next we shall estimate  $\|v\|$ .

2.4.3. *Estimate of  $\|v\|$ .* We compute

$$\begin{aligned} & \frac{1}{2} \left( c \frac{d}{dy} + \frac{d^2}{dy^2} \right) \|v\|^2 - \langle cv_y + v_{yy}, v \rangle - \|v_y\|^2 \\ &= -v\hat{v}_y l'_1 + v\check{v}_y l'_2 - \bar{v}[\hat{\phi}_{l_1}(l'_1)^2 + \hat{\phi}_{l_2} l'_1 l'_2] - \bar{v}[\check{\phi}_{l_2} l'_1 l'_2 + \check{\phi}_{l_2}(l'_2)^2] \\ &= o(1)E_l[(l'_1)^2 + (l'_2)^2]. \end{aligned}$$

Here we have used

$$\hat{v}_y = -\hat{\phi}_x l'_1, \quad \check{v}_y = \check{\phi}_x l'_2.$$

Due to the non-degeneracy and stability property of  $g$  in  $\mathbb{R}$ , there holds

$$(2.27) \quad \|\psi_x\|^2 + \langle F''(\phi)\psi, \psi \rangle \geq 2\nu\|\psi\|^2 + |\tilde{\psi}|^2 + |\bar{\psi}|^2, \quad \forall \psi \perp \phi_{l_i}, \quad i = 1, 2$$

for some constant  $\nu > 0$  when  $2l = l_2 - l_1$  is sufficiently large. (See also [14] Lemma 6.3.)

Multiplying (2.22) by  $v$  and integrating on  $\mathbb{R}$ , we can obtain

$$\begin{aligned} \langle cv_y + v_{yy}, v \rangle &= \langle F'(v + \phi) - F'(\phi) - v_{xx}, v \rangle - \langle c\phi_y + \phi_{yy}, v \rangle \\ &\geq 2\nu(\|v\|^2 + |\tilde{v}|^2 + |\bar{v}|^2) - v\hat{\phi}_x - v\check{\phi}_x \\ &\quad - \langle \phi_{l_1 l_1}, v \rangle (l'_1)^2 - 2\langle \phi_{l_1 l_2}, v \rangle l'_1 l'_2 - \langle \phi_{l_2 l_2}, v \rangle (l'_2)^2 \\ &\geq \nu\|v\|^2 + O(1)E_l^2 + O(1)[(l'_1)^2 + (l'_2)^2]^2 \end{aligned}$$

for  $y > Y_1$  sufficiently large.

Hence, we derive

$$(2.28) \quad \frac{1}{2} \left( c \frac{d}{dy} + \frac{d^2}{dy^2} \right) \|v\|^2 - \nu\|v\|^2 \geq -M_3(E_l^2 + (l')^4), \quad y > Y_1$$

for some positive constant  $M_3$  sufficiently large.

Let  $\kappa_1 < 0 < \kappa_2$  be the two roots of the characteristic equation  $\kappa^2 + c\kappa - 2\nu = 0$  associated with the operator on the left hand side of (2.28). Hence, by the maximum principle for second order ordinary differential equations, we have

$$\|v\|^2 \leq M_1 e^{\kappa_1(y-Y_1)} + M_2 \int_{Y_1}^y (E_l^2 + (l')^4) e^{\kappa_1(z-y)} dz + M_2 \int_{Y_1}^y (E_l^2 + (l')^4) e^{\kappa_2(z-y)} dz$$

for some positive constants  $M_1, M_2$ . It is easy to see that

$$\frac{d}{dz} [(E_l^2 + (l')^4) e^{\frac{1}{2}\kappa_1(z-y)}] < 0, \quad y > Y_1,$$

and

$$\frac{d}{dz} [(E_l^2 + (l')^4) e^{\frac{1}{2}\kappa_2(z-y)}] > 0, \quad y > Y_1$$

when  $Y_1$  is sufficiently large.

Hence we derive

$$\begin{aligned} \|v\|^2 &\leq M_1 e^{\kappa_1 y} + M_2 \int_y^\infty [(E_l^2 + (l')^4) e^{\frac{1}{2}\kappa_1(z-y)}] \cdot e^{\frac{1}{2}\kappa_1(z-y)} dz \\ &\quad + M_2 \int_{Y_1}^y [(E_l^2 + (l')^4) e^{\frac{1}{2}\kappa_2(z-y)}] \cdot e^{\frac{1}{2}\kappa_2(z-y)} dz \\ &\leq M_1 e^{\kappa_1 y} + M_0 (E_l^2 + (l')^4) \end{aligned}$$

for some positive constant  $M_1, M_0$ .

2.4.4. *Derivation of asymptotic formula for  $2l = l_2 - l_1$ .* Now we can write (2.25), (2.26) as

$$(2.29) \quad \begin{cases} cl'_1 + l''_1 = -Ae^{-2\mu l(y)}(1 + o(1)) + o(1)(l'_2 - l'_1) + o(1)l''_2 \\ cl'_2 + l''_2 = Ae^{-2\mu l(y)}(1 + o(1)) + o(1)(l'_2 - l'_1) + o(1)l''_1. \end{cases}$$

From (2.29) we can deduce

$$(2.30) \quad (c + o(1))l' + l'' = (A + o(1))e^{-2\mu l(y)}.$$

As in [14], we can define  $Q(y) = e^{2\mu l(y)}$ , which satisfies

$$2\mu A + o(1) = (c + o(1))Q' + Q''.$$

Solving this equation explicitly, we obtain

$$Q'(y) = \frac{2\mu A}{c} + o(1), \quad Q(y) = \frac{2\mu A}{c}y + o(y).$$

Hence we derive

$$\begin{aligned} l(y) &= \frac{1}{2\mu} \ln(y) + \frac{1}{2\mu} \ln\left(\frac{2\mu A}{c}\right) + o(1) \\ l'(y) &= \frac{1 + o(1)}{2\mu y}, \quad E_l = \frac{c + o(1)}{2\mu A y}, \quad l''(y) = \frac{o(1)}{2\mu y} \\ 0 > l'_1(y) &\geq -2l'(y) = -\frac{1 + o(1)}{\mu y}, \quad 0 < l'_2(y) \leq 2l'(y) = \frac{1 + o(1)}{\mu y}. \end{aligned}$$

Hence, we obtain an explicit estimate for  $\|v\|$  in term of  $y$

$$(2.31) \quad \|v\|^2 \leq \frac{O(1)}{y^2}.$$

2.4.5. *Derivation of asymptotical formulas for  $l_i, i = 1, 2$ .* Now we shall examine more carefully the ordinary differential equations (2.25) and (2.26).

Define

$$w(x, y) := u(x, y) - g(x - l_1(y)), \quad (x, y) \in D := \{|x - l_1(y)| \leq l(y)/2, y \geq Y_1\}$$

Then  $w$  satisfies a similar equation as (2.22) in  $D$  with  $v$  replaced by  $w$ :

$$(2.32) \quad \begin{aligned} &w_{xx} + w_{yy} + cw_y - ((F'(w + g(x - l_1)) - F'(g(x - l_1))) \\ &\quad + g''(x - l_1)(l'_1)^2 - g'(x - l_1)(cl'_1 + l''_1)) = 0, \quad (x, y) \in D. \end{aligned}$$

It is also easy to see that

$$\|w(\cdot, \cdot - l_1)\|_{L^2([l_1-2, l_1+2])} \leq \|v\| + \|\phi - g(\cdot - l_1)\|_{L^2([l_1-2, l_1+2])} \leq \|v\| + O(1)E_l \leq O(1)y^{-1}.$$

and

$$\|w(\cdot, \cdot - l_1)\|_{L^p([l_1-2, l_1+2])} \leq O(1) \cdot y^{-1}$$

for any  $p > 2$ .

Then, for any fix  $y_0 > Y_1$ , using the standard  $L^p$  interior estimate of elliptic equations for (2.32) and the Sobolev inequality in  $[l_1(y_0) - 2, l_1(y_0) + 2] \times [y_0 - 2, y_0 + 2]$ , we obtain

$$\|\nabla w(\cdot, y)\|_{L^\infty([l_1-1, l_1+1])} \leq O(1) \cdot y^{-1}, \quad y \geq Y_1$$

and hence

$$\|\nabla v(\cdot, y)\|_{L^\infty([l_1-1, l_1+1])} + \|v(\cdot, y)\|_{L^\infty([l_1-1, l_1+1])} \leq O(1) \cdot y^{-1}, \quad y \geq Y_1.$$

Using the standard  $L^p$  interior estimate of elliptic equations for (2.22) outside  $\bar{\Gamma} := \{(x, y) : |x - l_1(y)| + |x - l_2(y)| < 1, y > Y_1\}$  as well as the above estimate in  $\bar{\Gamma}$ , we can also obtain

$$\|\nabla v(\cdot, y)\| + \|\nabla v(\cdot, y)\|_{L^\infty(\mathbb{R})} + \|v(\cdot, y)\|_{L^\infty(\mathbb{R})} \leq O(1) \cdot y^{-1}, \quad y \geq Y_1.$$

Now we use re-examine (2.4.2), (2.21), (2.23) and (2.24), and obtain

$$(2.33) \quad cl'_1 + l''_1 = -\frac{E_l}{e} + O(1) \cdot y^{-2}, \quad y > Y_1.$$

Similarly, we can derive

$$(2.34) \quad cl'_2 + l''_2 = \frac{E_l}{e} + O(1) \cdot y^{-2}, \quad y > Y_1.$$

Hence, we have

$$c(l_1 + l_2)' + (l_1 + l_2)'' = O(1) \cdot y^{-2}, \quad y > Y_1$$

and therefore

$$l_1 + l_2 = O(1) \cdot y^{-1}, \quad y > Y_1.$$

This leads to

$$(2.35) \quad \begin{cases} l_1(y) = -\frac{1}{2\mu} \ln(y) - \frac{1}{2\mu} \ln\left(\frac{2\mu A}{c}\right) + B + o(1) \\ l_2(y) = \frac{1}{2\mu} \ln(y) + \frac{1}{2\mu} \ln\left(\frac{2\mu A}{c}\right) + B + o(1) \end{cases}$$

for some constant  $B$ . Lemma 2.7 then follows directly with

$$C_1 = -\frac{1}{2\mu} \ln\left(\frac{2\mu A}{c}\right) + B, \quad C_2 = -\frac{1}{2\mu} \ln\left(\frac{2\mu A}{c}\right) + B.$$

**2.5. The moving plane procedure.** In this subsection, we shall use the moving plane method to finish the proof of Theorem 1.1. Due to the fact that the asymptotical behavior of  $u$  is not homogeneous near infinity, in particular, there is a transition layer along  $\Gamma$ , the classic moving plane method has to be carefully modified. Indeed, we have to use the exact asymptotical formulas of the 0-level sets  $x = k_i(y)$ ,  $i = 1, 2$  near infinity as well the asymptotical behavior of  $u$  along these curves.

Define  $u_\lambda(x, y) := u(2\lambda - x, y)$  and  $w_\lambda := u_\lambda - u$  in  $D_\lambda := \{(x, y) : x \geq \lambda, y \in \mathbb{R}\}$ .

**Lemma 2.9.** *When  $\lambda$  is sufficiently large, there holds  $w_\lambda > 0$  in  $D_\lambda$ .*

*Proof.* When  $\lambda > \lambda_0$  is sufficiently large, by Lemma 2.7 we know that

$$k_1^\lambda(y) := 2\lambda - k_1(y) \geq k_2(y), \quad \forall y \geq Y_1.$$

By Lemma 2.3 and Lemma 2.4, we see that there exist constants  $K > 0, Y_2 > Y_1$  and  $\lambda_1$  sufficiently large such that when  $\lambda > \lambda_1$ , there hold  $w_\lambda > 0$  in  $D_{K, Y_2, \lambda} := \{(x, y) \in D_\lambda : x < k_1^\lambda(y) + K, y \geq Y_2\}$  and  $u < \alpha^-$  in  $D_{K, Y_2, \lambda}^c := \{(x, y) \in D_\lambda : x > k_1^\lambda(y) + K, y \geq Y_2 \text{ or } \forall x \geq \lambda, y \leq Y_2\}$ . Note that  $F''(s) > \mu_0 > 0$  for  $s \in (-1, \alpha^-]$  by the definition of  $\alpha^-$ .

We claim that  $w_\lambda \geq 0$  in  $D_\lambda$  for  $\lambda > \lambda_1$ . If it is not true, there exists a sequence of points  $\{(x_m, y_m)\}_{m=1}^\infty \in D_{K, Y_2, \lambda}^c$  such that

$$\lim_{m \rightarrow \infty} w_\lambda(x_m, y_m) = \lim_{m \rightarrow \infty} (u_\lambda(x_m, y_m) - u(x_m, y_m)) = \inf_{D_{K, Y_2, \lambda}^c} w_\lambda(x, y) < 0.$$

It can be seen that  $u_\lambda(x_m, y_m) < \alpha^-$  when  $m$  is large enough. Then we can follow the standard translating arguments to obtain a contradiction. Define  $w_\lambda^m(x, y) := w_\lambda(x + x_m, y + y_m)$  in  $D_{K, Y_2, \lambda}^c - (x_m, y_m)$ . Then  $w_\lambda^m$  converges to  $w_\lambda^\infty(x, y)$  in  $C_{loc}^3(D^\infty)$  for some piecewise Lipschitz domain  $D^\infty$  in  $\mathbb{R}^2$  which contains a small ball centered at the origin. Furthermore,  $w_\lambda^\infty$  attains its negative minimum at the origin and satisfies a linearized equation

$$(2.36) \quad w_{xx} + w_{yy} + cw_y - F''(\xi(x, y))w = 0, \quad (x, y) \in D^\infty$$

where  $\xi(x, y) = su(x, y) + (1-s)u_\lambda(x, y)$  for some  $s \in (0, 1)$  and  $F''(\xi(0, 0)) > \mu_0 > 0$ . This is a contradiction, which leads to the claim. Then the lemma follows from the strong maximum principle (or the Harnack inequality) applied to an elliptic equation similar to (2.36) which is satisfied by  $w_\lambda$ .  $\square$

Now we define

$$\Lambda = \inf\{\lambda : u_\lambda(x, y) > u(x, y), (x, y) \in D_\lambda\}.$$

**Lemma 2.10.** *There holds*

$$\Lambda = (C_1 + C_2)/2$$

where  $C_1, C_2$  are as in Lemma 2.7.

*Proof.* We shall prove this lemma by contradiction. Suppose the lemma does not hold. By Lemma 2.3 and Lemma 2.7, we can easily see that  $\Lambda > (C_1 + C_2)/2$  and  $w_\Lambda > 0, \forall (x, y) \in D_\Lambda$ . Then there exists a sequence of numbers  $\{\lambda_m\}$  such that  $\lambda_m < \Lambda$ , and  $\lim_{m \rightarrow \infty} \lambda_m = \Lambda$  and the infimum of  $w_{\lambda_m}$  in  $D_{\lambda_m}$  is negative. Using Lemma 2.3, Lemma 2.4, Lemma 2.7 and the translating arguments in the proof of Lemma 2.9, we can show that the infimum of  $w_{\lambda_m}$  in  $D_{\lambda_m}$  is achieved at a point  $(x_m, y_m)$ , i.e.,

$$(2.37) \quad w_{\lambda_m}(x_m, y_m) = \inf_{D_{\lambda_m}} w_{\lambda_m} < 0.$$

Since  $w_{\lambda_m}$  satisfies an elliptic equation similar to (2.36) with  $\xi(x_m, y_m) = su(x_m, y_m) + (1-s)u_{\lambda_m}(x_m, y_m)$  for some  $s \in (0, 1)$ , by the strong maximum principle we know that  $u(x_m, y_m) > \alpha^-$  and hence  $y_m > -K_1$  and  $x_m - k_1(y_m) < K$  if  $y_m > Y_1$  for some constant  $K, K_1 > 0$  independent of  $m$ . By Lemma 2.3, Lemma 2.7 and the assumption  $\Lambda > (C_1 + C_2)/2$ , we know  $y_m < K_2$  for some constant  $K_2$  independent of  $m$ . Therefore there exists a subsequence of  $\{m\}$  (still denoted by the same) such



that  $(x_m, y_m)$  converges to  $(x_0, y_0) \in D_\Lambda$  and  $w_{\lambda_m}$  converges to  $w_\Lambda$  in  $C_{loc}^3(D_\Lambda)$  as well as in  $C^3(B_1(x_0, y_0) \cap \bar{D}_\Lambda)$ . It is easy to see that  $\frac{\partial}{\partial x} w_\Lambda(x_0, y_0) = 0$ . Furthermore,  $w_\Lambda$  satisfies an elliptic equation similar to (2.36) in  $D_\Lambda$ , hence  $(x_0, y_0)$  must be on the boundary of  $D_\Lambda$ . Then by the Hopf Lemma, we have  $\frac{\partial}{\partial x} w_\Lambda(x_0, y_0) < 0$ . This is a contradiction, which proves the lemma.  $\square$

We note that  $u_\Lambda \geq u$  in  $D_\Lambda$  and  $u_x(\lambda, y) = -\frac{1}{2} \frac{\partial}{\partial x} w_\lambda(\lambda, y) > 0, \forall y \in \mathbb{R}$  when  $\lambda > \Lambda$ . Similarly, we can use the moving plane method from the left, i.e., repeating the above procedure for  $w_\lambda :=$  in  $D_\lambda^- := \{(x, y) : x < \lambda\}$ , and conclude  $u_\Lambda \geq u$  in  $D_\Lambda^-$ . Therefore, Theorem 1.1 is proven.

The uniqueness of the traveling wave solutions (up to translation) still remains an open question.

### 3. CLASSIFICATION OF TRAVELING WAVE SOLUTIONS FOR THE UNBALANCED ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

In this section, we shall assume that the double well potential  $F$  in the Allen-Cahn equation (1.1) is unbalanced, i.e.,  $F$  satisfies (1.2) and  $F(1) > F(-1) = 0$ . In this case, one dimensional traveling wave solution  $g$  to (1.6) exists for a unique  $c_0 > 0$  which only depends on  $F$ , and  $g$  is unique up to translation. It is easy to see that a rotation of the trivial extension of  $g$  to two dimensional plane is also a traveling wave solution of (1.3) for some constant  $c$ . Indeed, if  $\alpha \neq \pi/2, 3\pi/2$ , then  $u(x, y) = g(y \cos \alpha - \sin \alpha)$  satisfies (1.3) with  $c = \frac{c_0}{\cos \alpha}$ . In addition to the one dimensional traveling wave solutions, so called  $V$ -shaped two dimensional traveling wave solutions are shown to exist in [30], [42]. These solutions are monotone in  $y$  and even with respect to  $x$  after a proper translation. The 0-level set of such solutions are asymptotically two straight rays forming a shape of  $V$ . The existence result may be stated as follows.

**Theorem B** (Hamel, Monneau, Roquejoffre [30]; Ninomiya, Taniguchi, [42]; 2005). *For each  $\alpha \in [0, \pi/2)$  there exists a solution  $u_\alpha$  of (1.3), (1.4) and (1.5) such that  $c = \frac{c_0}{\cos \alpha}$  and  $u_\alpha$  is even in  $x$  and decreasing in  $|x|$ . The 0-level set of  $u$  is a globally Lipschitz graph of  $y = k(x)$  and  $k(x) = (\tan \alpha + o(1))|x|$  as  $|x|$  goes to infinity.*

Furthermore, it is shown in [31] that such  $V$  shaped traveling wave solutions are unique for each  $\alpha \in [0, \pi/2)$ . Indeed, the following classification theorem is proven.

**Theorem C** (Hamel, Monneau, Roquejoffre, 2006). *Suppose  $u$  is a solution to (1.3), (1.4) and (1.5). Assume further that there exists a globally Lipschitz function  $\psi$  such that*

$$(3.1) \quad \begin{cases} \liminf_{A \rightarrow +\infty, y \geq A + \psi(x)} u(x, y) > 0, \\ \limsup_{A \rightarrow -\infty, y \leq A + \psi(x)} u(x, y) < 0. \end{cases}$$

*Then  $c \geq c_0$ , and  $u$  must be either planar, i.e.  $u(x, y) = g(y \cos \alpha \pm x \sin \alpha + b)$  with  $\alpha = \cos^{-1}(c_0/c) \in [0, \pi/2)$  and a constant  $b$ , or  $u$  is the unique even  $V$ -shaped traveling wave solution  $u_\alpha$  (up to translation in  $x$  and  $y$ ).*

We note that for  $n \geq 3$ , similar conic shaped solutions are also shown to exist in [30], and the uniqueness of traveling wave solutions with 0-level set being prescribed asymptotical circular cone is proven in [30]. More complicated pyramidal traveling

wave solutions also exist for  $n \geq 3$ , and these solutions are unique when the 0-level sets are prescribed as given pyramidal cones at infinity (see [45], [46]).

The above classification theorem is very interesting. However, the condition (3.1) is too restrictive. We shall show that this condition can indeed be dropped. For this purpose, it suffices to show that 0-level set of  $u$  must be global Lipschitz, since the 0-level set function  $y = k(x)$  can serve as the function  $\psi$  in (3.1).

**Lemma 3.1.** *Assume that  $u$  is a solution to (1.3), (1.4) and (1.5), and the graph of  $y = k(x)$  is the 0-level set of  $u$ . Then  $k(x) \in C^3(\mathbb{R})$  and  $|k'(x)| \leq C, x \in \mathbb{R}$  for some constant  $C > 0$ .*

*Proof.* It is easy to see that  $k(x)$  is in  $C^3(\mathbb{R})$ . We shall prove the global Lipschitz property by contradiction. Assume that there exists a sequence  $\{x_m\}$  such that  $k'(x_m) \rightarrow \infty$  as  $m$  tends to infinity. Since  $u_x(x, k(x)) + u_y(x, k(x))k'(x) = 0, \forall x \in \mathbb{R}$  and  $\nabla u$  is bounded in  $\mathbb{R}^2$ , we derive  $u_y(x_m, k(x_m)) \rightarrow 0$  as  $m$  goes to infinity. We shall investigate the translation of  $u$  along  $(x_m, k(x_m))$ . Define  $u^m(x, y) := u(x + x_m, y + y_m)$ . Since  $u$  is bounded in  $C^{3,\beta}(\mathbb{R}^2)$  for some  $\beta \in (0, 1)$ , it is easy to see that  $u^m$  (up to a subsequence) converges to  $u^*$  in  $C_{loc}^3(\mathbb{R}^2)$ , and  $u^*$  satisfies (1.3). Hence  $u_y^*(x, y)$  satisfies the linearized equation

$$(3.2) \quad w_{xx} + w_{yy} + cw_y - F''(u^*)w = 0, \quad (x, y) \in \mathbb{R}^2.$$

By (1.4), we know that  $u_y^*(x, y) \geq 0, \forall (x, y) \in \mathbb{R}^2$ . Since  $u_y^*(0, 0) = \lim_{m \rightarrow \infty} u_y(x_m, k(x_m)) = 0$ , by the strong maximum principle for elliptic equations we obtain  $u_y^* \equiv 0$  in  $\mathbb{R}^2$ .

Therefore,  $u^*(x, y) = u^*(x)$  satisfies the one dimensional stationary Allen-Cahn equation (2.5) with  $|u^*(x)| \leq 1, x \in \mathbb{R}$  and  $u^*(0) = 0$ . Then we have either

**Case I:**  $u^*(x) = g_\alpha(x \pm K_\alpha), x \in \mathbb{R}$ , where  $g_\alpha$  is a periodic solution of the one dimensional Allen-Cahn equation

$$(3.3) \quad \begin{cases} g_\alpha''(x) - F'(g_\alpha(x)) = 0, & x \in \mathbb{R}, \\ g_\alpha'(0) = 0, & g_\alpha(0) = \alpha, \end{cases}$$

with  $\alpha = \max_{\mathbb{R}} u^*(x) \geq 0$  and  $K_\alpha$  is the smallest positive zero of  $g_\alpha$  if  $\alpha > 0$ ; or

**Case II:**  $u^*(x) = g_*(x \pm K_*)$  where  $g_*$  satisfies

$$(3.4) \quad \begin{cases} g_*''(x) - F'(g_*(x)) = 0, & |g_*(x)| < 1, \quad x \in \mathbb{R}, \\ g_*'(0) = 0, & \lim_{|x| \rightarrow \infty} g_*(x) = 1 \end{cases}$$

with  $K_* > 0$  being the only positive zero of  $g_*$ .

We note that  $g_0 \equiv 0$  and there is no standing wave solution to (1.6) with  $c_0 = 0$  in this case. It is well-known that  $g_\alpha$  is unstable in the sense that the linearized operator

$$\mathcal{L}_\alpha \psi := -\psi'' + F''(g_\alpha)\psi$$

has a negative first eigenvalue  $-\mu_\alpha$  in the periodic subclass of  $H^2(\mathbb{R})$  with period  $L = L(\alpha)$ . It is also well-known that  $g_*$  is unstable in the sense that the linearized operator

$$\mathcal{L}_* \psi := -\psi'' + F''(g_*)\psi$$

has a negative first eigenvalue  $-\mu_*$  in  $H^2(\mathbb{R})$ . (See, e.g., [33].)

Now we repeat the computations as in (2.2) in the proof of Lemma 2.1., and obtain

$$(3.5) \quad h'(x) = F(1) - c \int_{\mathbb{R}} u_y^2 dy, \quad \forall x \in \mathbb{R}.$$

Hence, for any  $a < b$  we have

$$(3.6) \quad \int_a^b \int_{\mathbb{R}} u_y^2 dy dx = \frac{1}{c} (h(a) - h(b)) + \frac{F(1)}{c} (b - a).$$

On the other hand, as in the proof of Lemma 2.2 we define

$$\bar{h}_m(x) = \int_{-\infty}^0 \frac{\partial u^m}{\partial x} \frac{\partial u^m}{\partial y} dy.$$

It is easy to see that  $|\bar{h}_m(x)| < C$  for some constant  $C$  independent of  $x$  and  $m$ . We can also derive

$$\bar{h}'_m(x) = -c \int_{-\infty}^0 \left( \frac{\partial u^m}{\partial y} \right)^2 dy + \frac{1}{2} \left( \frac{\partial u^m}{\partial x} \right)^2(x, 0) - \frac{1}{2} \left( \frac{\partial u^m}{\partial y} \right)^2(x, 0) + F(u^m(x, 0)), \quad \forall x \in \mathbb{R}$$

In Case I, in view of (3.6) we have for any fix  $R > 0$ ,

$$\int_{-R}^R \left[ \frac{1}{2} \left( \frac{\partial u^m}{\partial x} \right)^2(x, 0) - \frac{1}{2} \left( \frac{\partial u^m}{\partial y} \right)^2(x, 0) + F(u^m(x, 0)) \right] dx < C + 2F(1)R$$

for some constant  $C$  independent of  $m, R$ .

Letting  $m$  go to infinity, from  $u^*(x) = g_\alpha(x - K_\alpha)$  we obtain

$$\int_{-R}^R \left[ \frac{1}{2} |g'_\alpha|^2 + F(g_\alpha) \right] dx \leq C + 2F(1)R.$$

However, by the property (1.2) of  $F$ , we have  $F(g_\alpha) \geq F(\alpha) > F(1)$  in  $\mathbb{R}$  and hence

$$\int_{-R}^R \left[ \frac{1}{2} |g'_\alpha|^2 + F(g_\alpha) \right] dx > 2F(\alpha)R.$$

This is a contradiction when  $R$  is sufficiently large.

In Case II, in view of 3.6 we have, for any fix  $R > 0$ ,

$$(3.7) \quad \begin{aligned} & \int_{-R}^R \left[ \frac{1}{2} \left( \frac{\partial u^m}{\partial x} \right)^2(x, 0) - \frac{1}{2} \left( \frac{\partial u^m}{\partial y} \right)^2(x, 0) + F(u^m(x, 0)) - F(1) \right] dx \\ & \leq (\bar{h}_m(R) - h(R + x_m)) - (\bar{h}_m(-R) - h(-R + x_m)). \end{aligned}$$

We note that for any  $x \in \mathbb{R}$ ,

$$|h(x + x_m) - \bar{h}_m(x)| \leq C \int_0^\infty u_y(x + x_m, k(x_m) + y) dy \leq C[1 - u(x + x_m, k(x_m))]$$

and hence

$$\lim_{x \rightarrow \pm\infty} |h(x + x_m) - \bar{h}_m(x)| \leq C[1 - u^*(x)]$$

for some constant  $C$ .

Since  $u^* = g_*$ , we have

$$\lim_{x \rightarrow \pm\infty} \left[ \lim_{m \rightarrow \infty} (h(x + x_m) - \bar{h}_m(x)) \right] = 0.$$

From 3.7, by first letting  $m$  go to infinity and then letting  $R$  go to infinity, we obtain

$$\int_{\mathbb{R}} \left[ \frac{1}{2} |g'_*|^2 + F(g_*) - F(1) \right] dx \leq 0.$$

On the other hand, it is easy to see

$$\frac{1}{2}|g'_*|^2 - F(g_*(x)) = -F(1), \quad \forall x \in \mathbb{R}.$$

and hence  $F(g_*(x)) - F(1) > 0, \forall x \in \mathbb{R}$ . This leads to a contradiction. Therefore, Lemma 3.1 is proven.  $\square$

Combining Lemma 3.1 with Theorem C, we immediately have the following classification theorem for traveling wave solutions of the unbalanced Allen-Cahn equation.

**Theorem 3.2.** *Assume that  $F$  is a unbalanced double well potential satisfying (1.2) and  $F(1) > F(-1) = 0$ . Suppose  $u$  is a solution to (1.3), (1.4) and (1.5). Then  $c \geq c_0$  where  $c_0$  is the unique speed of one dimensional traveling wave solution  $g$  as in (1.6), and  $u$  must be either planar, i.e.  $u(x, y) = g(y \cos \alpha \pm x \sin \alpha + b)$  with  $\alpha = \cos^{-1}(c_0/c) \in [0, \pi/2)$  and  $b$  being a constant, or  $u$  is the unique even  $V$ -shaped traveling wave solution  $u_\alpha$  (up to a translation in  $x$  and  $y$ ).*

#### 4. TRAVELING WAVES SOLUTIONS CONNECTING OTHER ONE DIMENSIONAL STATIONARY SOLUTIONS

If we drop the limit assumption (1.5) and instead define

$$u^\pm(x) = \lim_{y \rightarrow \pm\infty} u(x, y), \quad x \in \mathbb{R},$$

then there are eight possibilities for the balanced Allen-Cahn equation:

- (1)  $u^+ = g(x - K_1), \quad u^- \equiv -1;$
- (2)  $u^+ = g(K_1 - x), \quad u^- \equiv -1;$
- (3)  $u^+ \equiv 1, \quad u^- = g(x - K_2);$
- (4)  $u^+ \equiv 1, \quad u^- = g(K_2 - x);$
- (5)  $u^+ = g(x - K_1), \quad u^- = g(x - K_2), \quad K_1 < K_2;$
- (6)  $u^+ = g(K_1 - x), \quad u^- = g(K_2 - x), \quad K_1 > K_2;$
- (7)  $u^+ = g_\alpha(x - K), \quad u^- \equiv -1;$
- (8)  $u^+ \equiv 1, \quad u^- = g_\alpha(x - K),$

where  $g_\alpha(x), \alpha \in [0, 1)$  is the periodic solution of one dimensional Allen-Cahn equation (3.3) (note that  $g_0 \equiv 0$ ),  $K$  is some constant.

Modifying the arguments in the proofs of Lemma 2.1-2.5, we can exclude Cases (1)-(6) by showing similar properties for  $u$  as in Lemma 2.1-2.5. For example, to exclude Case (1), we modify (2.2) and (2.3) as follows.

$$\begin{aligned} h'(x) &= \int_{\mathbb{R}} (u_{xx}u_y + u_xu_{xy})dy \\ (4.1) \quad &= -c \int_{\mathbb{R}} u_y^2 dy + F(u^+) + \frac{1}{2}|u_x^+|^2, \quad \forall x \in \mathbb{R}. \end{aligned}$$

Then

$$(4.2) \quad \int_a^b \int_{\mathbb{R}} u_y^2 dy dx \leq \frac{1}{c} (h((a) - h(b) + \mathbf{e}).$$

and Lemma 2.1 still holds. The rest will be essentially the same except that the 0-level set is the graph of a function  $y = \gamma(x)$  which is only defined in  $(-\infty, K_1)$ . We just replace  $x \rightarrow \infty$  by  $x \rightarrow K_1$  in the appropriate places. In this case, we can show Lemma 2.6 and Lemma 2.7 as well. This leads to a contradiction with  $u^+$ . Case (2) can be similarly excluded. Cases (3)-(6) are similar up to Lemma 2.5., and can be excluded directly by using the Hamiltonian identity as in the proof of Lemma 2.6. The details are omitted and left to the reader.

Therefore, we have the following nonexistence theorem.

**Theorem 4.1.** *Assume that  $F$  is a balanced double well potential, i.e.,  $F$  satisfies (1.2) and  $F(1) = F(-1) = 0$ . Then there exists no solution to (1.3) and (1.4) with the limits being one of the above Cases (1)-(6), i.e., at least one of  $u^+, u^-$  being a reflection and/or translation of  $g$ .*

For the unbalanced Allen-Cahn equation, since  $g$  is not a stationary solution of (2.5), there are only four possibilities for  $u^+, u^-$ : in addition to Cases (7), (8) listed above, there are the following two more cases:

$$(9) \quad u^+ = g_*(x - K), \quad u^- \equiv -1;$$

$$(10) \quad u^+ \equiv 1, \quad u^- = g_*(x - K),$$

where  $g_*$  is the unique solution to (3.4).

In Cases (7) and (8) there is no difference between the balanced and unbalanced Allen-Cahn equation since Case (7) is only involved with  $F(u)$  when  $u \leq \alpha < 1$  and Case (8) is only involved with  $F(u)$  when  $u \geq -\alpha > -1$ . These cases are indeed of monostable type. Case (7) could happen for sufficiently large  $c > 0$ , as shown in [33] for  $\alpha \in (0, 1)$  and [32] for  $\alpha = 0$  (see also [28] for a Bunsen flame model and [41] for related results). To be more precise, it is proven in [33] that for  $L > L_{min} := 2\pi\sqrt{-F''(0)}$ , there exists a positive minimum speed  $c_L > c_0$  such that (1.3) has a  $L$ -periodic solution  $u_{c,L}(x, y) = u_{c,L}(x, y + L)$  satisfying the limit condition  $u^+ = g_{\alpha(L)}, u^- \equiv -1$  if and only if  $c \geq c_L$ . Here  $\alpha(L)$  can be uniquely determined so that the period of  $g_{\alpha(L)}$  is  $L$ . It is also shown in [33] that Case (9) can happen for sufficiently large  $c > 0$ . Indeed, there exists  $c_\infty > c_0$  such that there exists a solution to (1.3) and (1.4) with *uniform* limits  $u^+, u^-$  being in Case (9) if and only if  $c \geq c_\infty$ . (See Theorems 1.1 and 1.4 in [33].)

However, Cases (8) and (10) can be excluded by using a generalized Hamiltonian identity.

**Theorem 4.2.** *Assume that  $F$  is a double well potential, i.e.,  $F$  satisfies (1.2). Then there exists no solution to (1.3) and (1.4) with  $u^+ \equiv 1, u^- = g_\alpha(x - K)$  for any constants  $\alpha \in [0, \infty), K \in \mathbb{R}$ . When  $F$  is unbalanced, there exists no solution to (1.3) and (1.4) with  $u^+ \equiv 1, u^- = g_*(x - K)$  for any constant  $K$ .*

*Proof.* We just note that as in (4.1), there holds

$$(4.3) \quad \begin{aligned} h'(x) &= \int_{\mathbb{R}} (u_{xx}u_y + u_xu_{xy})dy \\ &= -c \int_{\mathbb{R}} u_y^2 dy + F(1) - F(u^-) - \frac{1}{2}|u_x^-|^2, \quad \forall x \in \mathbb{R}. \end{aligned}$$

In Case (8),  $F(u^-) \geq F(\alpha) > F(1)$ , then

$$(4.4) \quad \int_a^b \int_{\mathbb{R}} u_y^2 dy dx \leq \frac{1}{c} [h(a) - h(b) + (b-a)(F(1) - F(\alpha))].$$

This leads to a contradiction when  $b-a$  is chosen sufficiently large.

In Case (10), we have

$$\lim_{x \rightarrow \infty} h(x) = \lim_{x \rightarrow -\infty} h(x) = 0.$$

Then

$$(4.5) \quad c \int_{\mathbb{R}} \int_{\mathbb{R}} u_y^2 dy dx \leq \int_{\mathbb{R}} (F(1) - F(g_*) - \frac{1}{2}|g_*'|^2) dx < 0.$$

This is a contradiction. The theorem is proven.  $\square$

We remark that Cases (8) and (10) could happen when the speed  $c$  is sufficiently negative. These cases are similar to (7) or (9) except that the traveling directions should be reversed. Another way to understand these cases is to reverse the spatial direction  $y$  while the speed  $c$  is kept positive. However, the monotone condition (1.4) is changed to decreasing in this approach.

Next, we shall show that when  $c > 0$  is sufficiently small, there is no monotone traveling wave solutions with limits as in Case (7) without requiring solutions being periodic in  $x$  nor the limits being uniform in  $x \in \mathbb{R}$ .

**Theorem 4.3.** *Assume that  $F$  is a double well potential, i.e.,  $F$  satisfies (1.2). Then, for  $L \in [0, \infty)$ , there exists a constant  $c_L^* > 0$  such that (1.3) and (1.4) has no solution with limits  $u^+ = g_{\alpha(L)}$ ,  $u^- \equiv -1$  when  $c < c_L^*$ , where  $g_{\alpha(L)}$  is the solution of (3.3) with a period  $L$  (we use the convention that  $\alpha(0) = 0$ ). Similarly there exists a constant  $c^* > 0$  such that (1.3) and (1.4) has no solution with limits  $u^+ = g_*$ ,  $u^- \equiv -1$  when  $c < c^*$  where  $g_*$  is the solution of (3.4).*

*Proof.* We shall first state a gradient estimate for general traveling wave solutions to (1.3) as in [40], where the same estimate is proved for stationary solutions to (1.11).

**Proposition 4.4.** *Assume that  $F(s) \geq 0, \forall s \in [-1, 1]$ . Suppose that  $u$  is a solution to (1.3). Then*

$$(4.6) \quad |\nabla u|^2(x, y) \leq 2F(u(x, y)), \quad (x, y) \in \mathbb{R}^n.$$

This inequality can be proven as in [40] with minor modifications. The proof is omitted here. The reader is referred to [24] for a complete proof.

Now suppose  $u$  is a solution to (1.3) and (1.4) with limits  $u^+ = g_{\alpha}$ ,  $u^- \equiv -1$ . Then, as in (4.1) we have

$$(4.7) \quad \begin{aligned} h'(x) &= \int_{\mathbb{R}} (u_{xx}u_y + u_x u_{xy}) dy \\ &= -c \int_{\mathbb{R}} u_y^2 dy + F(g_{\alpha}) + \frac{1}{2}|g'_{\alpha}|^2, \quad \forall x \in \mathbb{R}. \end{aligned}$$

Hence, in view of the fact  $F(g_{\alpha}(x)) \geq F(\alpha), \forall x \in \mathbb{R}$ , we obtain

$$(4.8) \quad c \int_a^b \int_{\mathbb{R}} u_y^2 dy dx \leq h(a) - h(b) + F(\alpha)(b-a).$$

On the other hand, by (4.6) there holds

$$(4.9) \quad \int_{\mathbb{R}} u_y^2 dy \leq \int_{\mathbb{R}} u_y \sqrt{2F(u)} dy \leq G(\beta)$$

where  $\beta := \inf_{x \in \mathbb{R}} g_\alpha(x) \in (-1, 0)$  with  $F(\beta) = F(\alpha)$ , and

$$G(s) := \int_{-1}^s \sqrt{2F(t)} dt > 0, \quad \forall s \in (-1, 1].$$

Hence,

$$c > \frac{F(\alpha)}{G(\beta)} > 0$$

and

$$(4.10) \quad c_L^* \geq \frac{F(\alpha(L))}{G(\beta(L))} > 0.$$

Similarly, if  $u$  is a solution to (1.3) and (1.4) with limits  $u^+ = g_*$ ,  $u^- \equiv -1$ , as in (4.7) we have

$$(4.11) \quad \begin{aligned} h'(x) &= \int_{\mathbb{R}} (u_{xx}u_y + u_x u_{xy}) dy \\ &= -c \int_{\mathbb{R}} u_y^2 dy + F(g_*) + \frac{1}{2}|g'_*|^2, \quad \forall x \in \mathbb{R}. \end{aligned}$$

Hence, in view of  $F(g_*(x)) \geq F(1), \forall x \in \mathbb{R}$ , we obtain

$$(4.12) \quad c \int_a^b \int_{\mathbb{R}} u_y^2 dy dx \geq h(a) - h(b) + F(1)(b - a).$$

On the other hand, by (4.6) there holds

$$(4.13) \quad \lim_{x \rightarrow \infty} \int_{\mathbb{R}} u_y^2 dy \leq \lim_{x \rightarrow \infty} \int_{\mathbb{R}} u_y \sqrt{2F(u)} dy \leq G(1) = \mathbf{e}.$$

Hence,

$$(4.14) \quad c \geq \frac{F(1)}{\mathbf{e}} > 0$$

and

$$(4.15) \quad c^* \geq \frac{F(1)}{\mathbf{e}} > 0.$$

The theorem is proven. □

The lower estimates of  $c_L^*$  and  $c_*$  above are obviously not optimal. Finally, we would like to ask the following questions.

**Open Questions.** *Regarding Case (7), is it true that  $c_L = c_L^*$ ? When  $c \geq c_L$ , are all solutions to (1.3) and (1.4) with limits  $u^+ = g_{\alpha(L)}$ ,  $u^- \equiv -1$  periodic? Regarding Case (9) for unbalanced  $F$ , is it true that  $c_\infty = c^*$ ? When  $c \geq c_\infty$ , are all solutions to (1.3) and (1.4) with limits  $u^+ = g_*$ ,  $u^- \equiv -1$  even in  $x$  after a proper translation in  $x$ ?*

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